QUANTUM THEORY OF THE DIELECTRIC CONSTANT OF A MAGNETIZED PLASMA AND ASTROPHYSICAL APPLICATIONS

I. *Theory*

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Abstract. A quantum mechanical treatment of an electron plasma in a constant and homogeneous magnetic field is considered, with the aim of (a) defining the range of validity of the magnetoionic theory (b) studying the deviations from this theory, in applications involving high densities, and intense magnetic field. While treating the magnetic field exactly, a perturbation approach in the photon field is used to derive general expressions for the dielectric tensor $\varepsilon_{\alpha\beta}$. The properties of $\varepsilon_{\alpha\beta}$ are explored in the various limits. Numerical estimates on the range of applicability of the magnetoionic theory are given for the case of the 'one-dimensional' electron gas, where only the lowest Landau level is occupied.

1. Introduction

The recent discovery of pulsars has triggered a series of papers [1] dealing with the properties of an electron gas imbedded in a superstrong magnetic field ($\approx 10^{12}$ G). Such fields had no existing proof other than the flux conservation law which dictates fields of the order of 10^6 G for white dwarfs and 10^{13} G for neutron stars [2] (see, however, newly proposed mechanism LOFER [3]). Very recently, fields of 10^6 G have indeed been found in white dwarfs although the statistics are still rather poor [4]. It seems, therefore, that whatever the creation mechanism will eventually turn out to be, such enormous fields have great probability of being really attached to the collapsed bodies such as neutron stars. In a series of papers, one of the authors has investigated the modifications induced in many physical processes by such fields [5]. Because of the existence of plasma, the computation of many of those processes calls for the use of the dielectric constant of the medium [6]. A plethora of formulae exist for such a dielectric constant, dealing mostly with the classical aspect of the problem [7]. The formula most used for $\varepsilon_{\alpha\beta}$ is the one given by the so-called magnetoionic theory which, in its derivation, is entirely classical [8]. The quantum mechanical treatment makes use of the density matrix or of the Wigner distribution function and Boltzmann equation [9]. Since the quantum theory of transport processes is a subtle subject not totally understood at the moment, we felt that there was the possibility of having missed some quantum mechanical properties by just adapting an equation that is essentially clas-

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sical. Besides, we wanted to investigate under what conditions the magnetoionic theory is reproduced by the quantum mechanical treatment.

After obtaining the quantum mechanical formulae for the dielectric constant, we will show that quite independently of the specific form of the distribution function, the long wavelength approximation reproduces the results of the magnetoionic theory. This puts an end the to question, indeed asked many times, as to whether the degeneracy of electrons often encountered in problems related to pulsars is properly taken into account by the magnetoionic formulae. We also recover the classical polarization tensor $\Pi_{ij}(k,\omega)$ in the limit of large quantum numbers. We want to emphasize that our treatment does not rely on the Boltzmann equation or Wigner distribution function. It follows from the very quantum mechanical definition of $\varepsilon_{\alpha\beta}$.

2. Wave Function and Eigenvalues of an Electron in a Magnetic Field

For the sake of completeness, we shall briefly review here the quantum mechanical properties of an electron in a constant and uniform magnetic field H. (For a detailed study see [5, 10].)

In the presence of such a field, electrons are trapped in quantum orbits which emerge from the Schroedinger Hamiltonian

$$\mathscr{H} = \frac{\boldsymbol{\pi}^2}{2M}; \qquad \boldsymbol{\pi} \equiv \mathbf{p} + \frac{e}{c}\mathbf{A}_0 .$$

Taking the z-direction to coincide with that of $\mathbf{H}$, one has (Landau gauge) $\mathbf{A}_0 = (-yH, 0, 0)$. The Hamiltonian and its eigenstates then take the form

$$\mathscr{H} = \frac{1}{2M}\left[(p_x - M\omega_H y)^2 + p_y^2 + p_z^2)\right], \tag{1}$$

$$\phi_{np_xp_z} = w_n(y - a^2 p_x/\hbar)\exp\left[i(p_x x + p_z z)/\hbar\right],$$

where $a \equiv (\hbar/M\omega_H)^{\frac{1}{2}}$; $\omega_H \equiv eH/Mc$ is the cyclotron frequency; $w_n(u)$ are the normalized harmonic oscillator wave functions, i.e.,

$$w_n(u) = (a\sqrt{\pi}2^n n!)^{-1/2}\exp\left[-u^2/2a^2\right] H_n(u/a), \tag{2}$$

where H_n stand for the Hermite polynomials [11]. The Hermite polynomials of Equation (2) relate to the one defined in this reference by the equation $H_n(\kappa) = 2^{n/2} He_n(\kappa\sqrt{2})$.

The energy spectrum is correspondingly given by

$$E_n(p_z) = \hbar\omega_H(n + \tfrac{1}{2}) + \frac{p_z^2}{2M}. \tag{3}$$

The momenta π_x and π_y are not constants of the motion, as one can easily check that

$$[\pi_x, \pi_y] = -i M\hbar\omega_H, \quad [\pi_\perp^2, \pi_\pm] = \pm 2(M\hbar\omega_H)\pi_\pm, \tag{4}$$

Where, $\pi_\pm \equiv \pi_x \pm i\pi_y$ act as raising and lowering operators

$$\pi_\pm |np_xp_z\rangle = C_n^{(\pm)} |n \pm 1 p_xp_z\rangle \tag{5}$$

with

$$C_n^{(+)} \equiv -\sqrt{2M\hbar\omega_H}\,(n+1)^{1/2}, \qquad C_n^{(-)} \equiv -\sqrt{2M\hbar\omega_H}\,n^{1/2}$$

3. The Dielectric Tensor

An electromagnetic wave of frequency ω, applied on a plasma, gives rise to induced charge and current densities which can either sustain, in a self-consistent way, or damp the propagation of the wave. Thus, the plasma will be transparent to the frequency ω, if the set of Maxwell's equations (or equivalently, the wave equation) accepts a solution at this frequency.

For a particular Fourier component ($\mathbf{k}$, ω), the wave equation is [12]

$$\left(k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\,\varepsilon_{\alpha\beta}\right) E_\beta = 0. \tag{6}$$

The dielectric properties of the medium are contained in the tensor $\varepsilon_{\alpha\beta}$ known as the dielectric tensor. The induced current density is given in terms of $\varepsilon_{\alpha\beta}$ as [12]

$$j_\alpha = i\,\frac{\omega}{4\pi}\,(\delta_{\alpha\beta} - \varepsilon_{\alpha\beta})\,E_\beta\,, \tag{6a}$$

as may be easily be proved from Maxwell's equation

$$(\boldsymbol{\nabla} \times \mathbf{B})_\alpha = \frac{4\pi}{c}\,j_\alpha - i\,\frac{\omega}{c}\,E_\alpha \equiv -\,i\,\frac{\omega}{c}\,\varepsilon_{\alpha\beta}E_\beta\,;$$

while the induced charge density is determined from the continuity equation, $\varrho = \mathbf{k}\cdot\mathbf{j}/\omega$.

Equation (6) has the form of an eigenvalue problem having solutions only for those values of $k^2c^2/\omega^2 = n^2$ which make the determinant vanish – i.e.,

$$\det\left[n^2(\delta_{\alpha\beta} - n_\alpha n_\beta) - \varepsilon_{\alpha\beta}\right] = 0\,, \tag{7}$$

where $n_\alpha = k_\alpha/|k|$. To each positive definite solution of the dispersion Equation (7), there is a corresponding wave which propagates with a refractive index $n = kc/\omega$.

The dispersive properties of the medium thus depend on the precise form of $\varepsilon_{\alpha\beta}$. In what follows, seeking the dielectric response of a magnetized quantum plasma to an electromagnetic perturbation ($\mathbf{A}$,ϕ), we will choose a gauge such that the scalar potential $\phi = 0$, and, therefore, $\mathbf{E} = i(\omega/c)\mathbf{A}$. The vector potential will, therefore, obey Equation (6) just like $\mathbf{E}$.

To derive a quantum mechanical expression for $\varepsilon_{\alpha\beta}$, we shall now evaluate, in perturbation theory, the ($\mathbf{k}$, ω) Fourier component of the current induced by the external field

$$\mathbf{A}(t) = \mathbf{A}\exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right] + \text{c.c.}$$

The current operator in second quantized form is

$$\mathbf{j}(\mathbf{r}, t) = -\frac{e}{M}\psi^{\dagger}(\mathbf{r})\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_0 + \frac{e}{c}\mathbf{A}(t)\right)\psi(\mathbf{r}) + \text{c.c.} \tag{8}$$

The unperturbed Hamiltonian is

$$\mathscr{H}_0 = \frac{1}{2M}\int \mathrm{d}^3 r \psi^{\dagger}(\mathbf{r})\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_0\right)^2 \psi(\mathbf{r}) = \sum_n E_n a_n^{\dagger} a_n ,$$

where a_n is the destruction operator for an electron of the corresponding state in the Landau spectrum, Equation (1). The operators a_n obey the usual anticommutation rules, and the expansion $\psi(r) = \sum_n \phi_n(\mathbf{r}) a_n$ was implied in the above equation.

Treating $\mathbf{A}(t)$ as a small perturbation, we write the interaction as

$$\mathscr{H}_I = \frac{e}{Mc}\int \mathrm{d}^3 r \psi^{\dagger}(\mathbf{r})\,(\boldsymbol{\pi}\cdot\mathbf{A}(t))\,\psi(\mathbf{r}) \cong$$

$$\cong \frac{e}{Mc}\sum_{nm}\left[\langle n|\,\boldsymbol{\pi}^k\cdot\mathbf{A}\,|m\rangle\, e^{-i\omega t} + \langle n|\,\boldsymbol{\pi}^{-k}\cdot\mathbf{A}\,|m\rangle\, e^{i\omega t}\right] a_n^{\dagger} a_m , \tag{9}$$

with

$$\boldsymbol{\pi}^k \equiv \tfrac{1}{2}\left(\exp[i\mathbf{k}\cdot\mathbf{r}]\,\boldsymbol{\pi} + \boldsymbol{\pi}\exp[i\mathbf{k}\cdot\mathbf{r}]\right). \tag{10}$$

Let the state vector ϕ_{NO} represent a quantum state of N electrons in the absence of external field, i.e., $H_0\phi_{NO} = E_{NO}\phi_{NO}$. In the presence of the perturbation, one has to first order in $\mathbf{A}$

$$\Phi_{NO} = \phi_{NO} + \phi_{NO}^{(+)} e^{-i\omega t} + \phi_{NO}^{(-)} e^{i\omega t}, \tag{11}$$

$$\phi_{NO}^{(+)} = \frac{e}{Mc}\sum_{nm}\frac{\langle n|\,\boldsymbol{\pi}^k\cdot\mathbf{A}\,|m\rangle}{E_{NO} - \mathscr{H}_0 + \hbar\omega}\, a_n^{\dagger} a_m \phi_{NO} =$$

$$= \frac{e}{Mc\hbar}\sum_{nm} a_n^{\dagger} a_m \phi_{NO}\,\frac{\langle n|\,\boldsymbol{\pi}^k\cdot\mathbf{A}\,|m\rangle}{\omega + \omega_{mn}}, \tag{11a}$$

$$\phi_{NO}^{(-)} \equiv -\frac{e}{Mc\hbar}\sum_{nm} a_m^{\dagger} a_n \phi_{NO}\,\frac{\langle m|\,\boldsymbol{\pi}^{-k}\cdot\mathbf{A}\,|n\rangle}{\omega - \omega_{mn}}, \tag{11b}$$

Where $\omega_{mn} = \hbar^{-1}(E_m - E_n)$, and $\langle m|\ \ |n\rangle$ is the matrix element between one-electron states.

It is easy to see that the resulting expression for the current is

$$\mathbf{j}(k, \omega) = \mathbf{j}^{(1)}(\mathbf{k}, \omega) + \mathbf{j}^{(2)}(\mathbf{k}, \omega) + \mathbf{j}^{(3)}(\mathbf{k}, \omega), \tag{12}$$

$$\mathbf{j}^{(1)}(\mathbf{k}, \omega) = -\frac{e^2}{Mc}\mathbf{A}\left(\phi_{NO}, \int \mathrm{d}^3 r \psi^{\dagger}\psi\phi_{NO}\right) =$$

$$= -\frac{e^2 N}{Mc}\mathbf{A},$$

$$j_\alpha^{(2)}(k,\omega) = -\frac{e^2}{M^2c}\left(\phi_{N0}, \int d^3r\psi^\dagger\pi_\alpha^{-k}\psi\phi_{N0}^{(+)}\right) =$$

$$= -\frac{e^2}{M^2c\hbar}\sum_{nmpq}\langle a_n^\dagger a_m a_p^\dagger a_q\rangle \frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle p|\boldsymbol{\pi}^k\cdot\mathbf{A}|q\rangle}{\omega-\omega_{pq}}.$$

$$j_\alpha^{(3)}(k,\omega) = -\frac{e^2}{M^2c}\left(\phi_{N0}^{(-)}, \int d^3r\psi^\dagger\pi_\alpha^{-k}\psi\phi_{N0}\right) =$$

$$= \frac{e^2}{M^2c\hbar}\sum_{nmpq}\langle a_n^\dagger a_m a_p^\dagger a_q\rangle \frac{\langle n|\boldsymbol{\pi}^k\cdot A|m\rangle\langle p|\pi_\alpha^{-k}|q\rangle}{\omega+\omega_{mn}}.$$

Comparing this expression with Equation (6a) we obtain the quantum mechanical expression for $\varepsilon_{\alpha\beta}$ as

$$\varepsilon_{\alpha\beta} = \left(1-\frac{\omega_p^2}{\omega^2}\right)\delta_{\alpha\beta} + \frac{\omega_p^2}{\omega^2}\tau_{\alpha\beta},$$

$$\tau_{\alpha\beta} \equiv \frac{-1}{NM\hbar}\sum_{nm}\left[\frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle m|\pi_\beta^k|n\rangle}{\omega-\omega_{mn}} - \frac{\langle n|\pi_\beta^k|m\rangle\langle m|\pi_\alpha^{-k}|n\rangle}{\omega+\omega_{mn}}\right]f_n(1-f_m), \tag{13}$$

where we have put $f_n \equiv \langle a_n^\dagger a_n\rangle$, $\omega_p^2 \equiv 4\pi Ne^2/M$.
By interchanging the dummy indices in the second term, Equation (13) can be recasted in the following more instructive form

$$\tau_{\alpha\beta} = \frac{-1}{NM\hbar}\sum_{nm}\frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle m|\pi_\beta^k|n\rangle}{\omega-\omega_{mn}}(f_n-f_m) = \tag{13a}$$

$$= \frac{-1}{NM\hbar}\sum_{nm}\left[\frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle m|\pi_\beta^k|n\rangle}{\omega-\omega_{mn}} - \frac{\langle n|\pi_\beta^k|m\rangle\langle m|\pi_\alpha^{-k}|n\rangle}{\omega+\omega_{mn}}\right]f_n. \tag{13b}$$

Either of the expressions (13a) and (13b) will be used on convenience in the following. It is worthwhile to notice that the statistical factor $(1-f_m)$ in Equation (13), which is a manifestation of the Pauli exclusion principle, has disappeared in (13b).

The matrix elements appearing in the expression for $\varepsilon_{\alpha\beta}$ may be evaluated by making use of the wave functions, Equation (1). It is easy to see that putting $k_x=0$

$$\langle n'p_x'p_z'|\pi_\alpha^{\boldsymbol{k}}|np_xp_z\rangle = \delta_{p_x}^{p'_x}\delta_{p_z}^{p'_z} + \hbar k_z\langle n'|\pi_\alpha^{k_y}|n\rangle, \tag{14}$$

where $\pi_\alpha^{k_y} \equiv \exp[ik_yy](\pi_\alpha + \frac{1}{2}\hbar k_\alpha)$. By making use of Equation (5) one can cast the resulting integrals in the form [13]

$$\langle m|e^{ik_y y}|\,n\rangle = \frac{1}{a}\int_{-\infty}^{\infty} \mathrm{d}u w_m^*(u)\, w_n(u) \exp[ik_y y] = C_{mn} I_{mn}(\tfrac{1}{2}a^2 k_y^2), \tag{15}$$

where

$$u \equiv y - \frac{a^2 p_x}{\hbar}; \qquad C_{mn} \equiv i^{m-n} \exp[ia^2 k\hbar^{-1} p_x]$$
$$I_{mn}(\varrho) \equiv (m!n!)^{-1/2} \exp[-\varrho/2]\, \varrho^{(m-n)/2} Q_n^{m-n}(\varrho), \tag{16}$$

and $Q_m^l(\varrho)$ are the Laguerre polynomials. The functions $I_{mn}(\varrho)$ of Equation (16) form an orthonormal set and have been studied Extensively in the literature [14].

One finds

$$\begin{aligned}
\langle m|\pi_x^{k_\perp}|\,n\rangle &= -i(\tfrac{1}{2}M\hbar\omega_H)^{1/2}\, C_{mn} I_{mn}^{(-)},\\
\langle m|\pi_y^{k_\perp}|\,n\rangle &= (\tfrac{1}{2}M\hbar\omega_H)^{1/2}\, C_{mn}\left(I_{mn}^{(+)} + \frac{ak_\perp}{\sqrt{2}} I_{mn}\right),\\
\langle m|\pi_z^{k_\perp}|\,n\rangle &= (p + \tfrac{1}{2}\hbar k_z)\, C_{mn} I_{mn},
\end{aligned} \tag{17}$$

where

$$I_{mn} \equiv I_{mn}(\tfrac{1}{2}a^2 k_\perp^2), \quad I_{mn}^{(\pm)} = \sqrt{n} I_{m,n-1} \pm \sqrt{n+1} I_{m,n+1}, \quad \text{and} \quad k_\perp \equiv k_y.$$

4. Statistical Averaging

It is now time to examine more closely the nature of the summations $\sum_{mn}$ and of the statistical quantities f_n appearing in Equation (13). The quantum states may be defined by the quantum numbers n, p_z, p_x of Equation (1) and the spin quantum number, $s = \pm 1$. The inclusion of the spin is straightforward in our calculation. It modifies the spectrum Equation (3) by an additive quantity, $\pm\frac{1}{2}\hbar\omega_H$, making all but the lowest Landau levels doubly degenerate. The modified spectrum is

$$E_{np} = \frac{p^2}{2M} + \hbar\omega_H n,$$

where $p \equiv p_z$. The level degeneracy due to spin is

$$a_n = 1 \quad \text{for} \quad n = 0,$$
$$a_n = 2 \quad \text{for} \quad n > 0,$$

while the additional degeneracy in the quantum number p_x has been discussed in the literature [15].

The density of electrons is expressed by

$$N = \frac{1}{V}\sum_{np_xps} f_{np_xps} = \frac{M\omega_H}{(2\pi\hbar)^2}\sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} \mathrm{d}p f(E_{np}), \tag{18}$$

where V is a macroscopic normalization volume.

Turning again to Equations (13), we note that the summation over p', p'_x, s' of the intermediate state $|m\rangle \equiv |m, p', p'_x, s'\rangle$ becomes trivial because of the δ-functions in the matrix element Equation (14). The corresponding conservation laws are $p' = p \pm \pm \hbar k_z$, $p'_x = p_x$, $s' = s$, the upper and lower signs refer to the first and second terms in Equation (13b). We may therefore write Equation (13a) and (13b) as

$$\tau_{\alpha\beta} = \frac{-1}{N} \sum_{mnpp_xs} \frac{(t_{\alpha\beta})_{mn}}{\omega - \omega_{mn}(p, p + \hbar k_z)} [f(E_{np}) - f(E_{mp'})], \tag{19}$$

$$= \frac{-1}{N} \sum_{mnpp_xs} \left[\frac{(t_{\alpha\beta})_{mn}}{\omega - \omega_{mn}(p, p + \hbar k_z)} - \frac{(t^{(1)}_{\alpha\beta})_{mn}}{\omega + \omega_{mn}(p, p - \hbar k_z)}\right] f(E_{np}), \tag{20}$$

with

$$(t_{\alpha\beta})_{mn} = \frac{1}{M\hbar} \langle n| \pi_\alpha^{-k_\perp} |m\rangle \langle m| \pi_\beta^{k_\perp} |n\rangle ,$$

$$(t^{(1)}_{\alpha\beta})_{mn} = \frac{1}{M\hbar} \langle n| \pi_\beta^{k_\perp} |m\rangle \langle m| \pi_\alpha^{-k_\perp} |n\rangle ,$$

$$\omega_{mn}(p, p \pm \hbar k_z) = (m - n)\, \omega_H \pm \frac{p}{M} k_z + \frac{\hbar k_z^2}{2M}.$$

Substituting Equations (17) for the matrix elements, we find for the components of the tensor $t_{\alpha\beta}$ the expressions

$$t_{xx} = \frac{\omega_H}{2} |I^{(-)}_{mn}|^2, \qquad t_{yy} = \frac{\omega_H}{2}\left(I^{(+)}_{mn} + \frac{ak_\perp}{\sqrt{2}} I_{mn}\right)^2,$$

$$t_{xy} = i\,\frac{\omega_H}{2}\left(I^{(+)}_{mn} + \frac{ak_\perp}{\sqrt{2}} I_{mn}\right) I^{(-)}_{mn}, \qquad t_{yx} = - t_{xy},$$

$$t_{zz} = \frac{1}{M\hbar} (p + \tfrac{1}{2}\hbar k_z)^2\, I^2_{mn}, \tag{21}$$

$$t_{zx} = - t_{xz} = - i \left(\frac{\omega_H}{2M\hbar}\right)^{1/2} (p + \tfrac{1}{2}\hbar k_z)\, I_{mn} I^{(-)}_{mn},$$

$$t_{zy} = t_{yz} = \left(\frac{\omega_H}{2M\hbar}\right)^{1/2} (p + \tfrac{1}{2}\hbar k_z)\, I_{mn}\left(I^{(+)}_{mn} + \frac{ak_\perp}{\sqrt{2}} I_{mn}\right).$$

The argument in the functions I_{mn} is $\frac{1}{2}a^2k_\perp^2$ as before. Similar expressions may be easily written for the tensor $t^{(1)}_{\alpha\beta}$

The integral over p in Equations (19) and (20) is performed under the analyticity convention [16],

$$\lim_{\eta \to 0} \frac{1}{\omega \pm \omega_{mn} + i\eta} = P \frac{1}{\omega \pm \omega_{mn}} - i\pi\delta(\omega \pm \omega_{mn}). \tag{22}$$

Thus the principal value of the integral in question will give rise to the Hermitian (refractive) part of the tensor $\tau_{\alpha\beta}$, while the integral over the δ-function gives the antihermitian (absorptive) part.

The case of propagation along the magnetic field ($\mathbf{k}\|\mathbf{H}$) is of particular interest because it brings out simple properties of the medium. In the following paragraph we shall give a detailed study of absorption and refraction in this direction.

5. Longitudinal Propagation k| |H

In this case ($k_\perp = 0$), one has

$$I_{mn}(0) = \delta_{mn}, \quad I_{mn}^{(\pm)}(0) = \sqrt{n}\delta_{n-1}^m \pm \sqrt{n+1}\delta_{n+1}^m .$$

Equations (21) then give $t_{xx} = t_{yy}$, $t_{xy} = -t_{yx}$ so that we can express all four components by introducing the linearly independent quantities

$$t_\pm = t_{xx} \mp i t_{xy} = \begin{cases} n\omega_H \delta_{n-1}^m \\ (n+1)\,\omega_H \delta_{n+1}^m \end{cases}$$

$$t_{zz} = \frac{1}{M\hbar}\left(p + \tfrac{1}{2}\hbar k_z\right)^2 \delta_n^m, \tag{23a}$$

$$t_{zx} = t_{xz} = 0, \quad t_{zy} = t_{yz} = 0.$$

Similarly, we find

$$t_\pm^{(1)} \equiv t_{xx}^{(1)} \mp i t_{xy}^{(1)} = \begin{cases} (n+1)\,\omega_H \delta_{n+1}^m \\ n\omega_H \delta_{n-1}^m \end{cases} \tag{23b}$$

$$t_{zz}^{(1)} = t_{zz}.$$

Accordingly, we find from Equation (20)

$$\tau_\pm = \mp \frac{\omega_H}{P_0} \sum_n a_n \int_{-\infty}^{\infty} \mathrm{d}p \left[\frac{n}{\omega_\pm - \dfrac{pk}{M} \mp \dfrac{\hbar k^2}{2M}} - \frac{n+1}{\omega_\pm - \dfrac{pk}{M} \pm \dfrac{\hbar k^2}{2M}} \right] f(E_{np}), \tag{24a}$$

$$\tau_{zz} = -\frac{1}{M\hbar P_0} \sum_n a_n \int_{-\infty}^{\infty} \mathrm{d}p \left[\frac{\left(p + \frac{1}{2}\hbar k\right)^2}{\omega - \dfrac{pk}{M} - \dfrac{\hbar k^2}{2M}} - \frac{\left(p - \frac{1}{2}\hbar k\right)^2}{\omega - \dfrac{pk}{M} + \dfrac{\hbar k^2}{2M}} \right] f(E_{np}), \tag{24b}$$

where $\omega_\pm \equiv \omega \pm \omega_H$. We have also introduced (*cf.* Equation (18)) the quantity

$$P_0 \equiv \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} \mathrm{d}p f(E_{np}) = N \frac{(2\pi\hbar)^2}{M\omega_H}.$$

The principal value of the above integrals gives the refractive part of $\tau_{\alpha\beta}$ in accordance with Equation (22). For the absorptive part we note that the integral over the δ-function in (19) gives quite generally

$$\tau_{\alpha\beta}^{A} = \frac{i\pi}{P_0}\frac{M}{k_z}\sum_{mn} a_n (t_{\alpha\beta})_{mn}\left[f(E_{np_l}) - f(E_{np_l} + \hbar\omega)\right], \tag{25}$$

with

$$p_l = \frac{M}{k_z}\left[\omega - l\omega_H\right] - \tfrac{1}{2}\hbar k_z, \quad l = m - n.$$

Substituting (23a) we find, accordingly.

$$\begin{aligned} \tau_{\pm}^{A} &= i\pi\frac{M\omega_H}{kP_0}\sum_n a_n \begin{Bmatrix} n \\ n+1 \end{Bmatrix}\left[f(E_{np_\pm}) - f(E_{np_\pm} + \hbar\omega)\right], \\ \tau_{zz}^{A} &= i\pi\frac{M^2\omega^2}{\hbar k^3 P_0}\sum_n a_n\left[f(E_{npo}) - f(E_{npo} + \hbar\omega)\right], \end{aligned} \tag{25a}$$

where

$$p_\pm = k^{-1}M\omega_\pm \mp \tfrac{1}{2}\hbar k, \qquad p_0 = k^{-1}M\omega - \tfrac{1}{2}\hbar k.$$

6. Special Cases

A. CLASSICAL LIMIT

In the limit of large quantum numbers, $\langle n\rangle \gg 1$, the discrete spectrum approaches a continuum. The quantum mechanical formulae then lead to the corresponding classical expressions by the following procedure [15]:

$$\sum_{npp_xs} = \frac{M\omega_H}{(2\pi\hbar)^2}\sum_n a_n \int_{-\infty}^{\infty} \mathrm{d}p \to \frac{2M^2\omega_H}{(2\pi\hbar)^2}\int_0^{\infty}\mathrm{d}n\int_{-\infty}^{\infty}\mathrm{d}v_z.$$

Now $2M^{-1}\hbar\omega_H n \cong v_\perp^2 = v_x^2 + v_y^2$ so that $\int \mathrm{d}v_\perp^2 = 1/\pi\int\int \mathrm{d}v_x\,\mathrm{d}v_y$ and, therefore,

$$\frac{M\omega_H}{(2\pi\hbar)^2}\sum_n a_n\int_{-\infty}^{\infty}\mathrm{d}p \to \frac{2M^3}{(2\pi\hbar)^3}\int \mathrm{d}^3v. \tag{26}$$

Equation (18) correspondingly becomes $N = 2M^3(2\pi\hbar)^{-3}\int \mathrm{d}^3v\, f(\mathbf{v})$. In rewriting Equation (19) we have

$$\begin{aligned} E_{mp'} - E_{np} \equiv \Delta E &= \hbar\left[l\omega_H + k_z v_z\left(1 + \frac{\hbar k_z}{2Mv_z}\right)\right] \cong \\ &\cong \hbar\left[l\omega_H + k_z v_z\right], \end{aligned}$$

where

$$l \equiv m - n, \qquad \mathbf{v} \equiv \boldsymbol{\pi}/M, \qquad p' = p + \hbar k_z.$$

Assuming that $l \ll n$, $\hbar k_z \ll p$, we may write

$$f(E_{mp'}) - f(E_{np}) \cong \Delta f = \frac{\hbar}{M}\left(\frac{l\omega_H}{v_\perp}\frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z}\right);$$

consequently, Equation (19) becomes

$$\tau_{\alpha\beta} = \frac{2M^3}{N(2\pi\hbar)^3} \sum_{l=-\infty}^{\infty} \int \mathrm{d}^3 v \left(\frac{l\omega_H}{v_\perp}\frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z}\right) \frac{\frac{\hbar}{M} t_{\alpha\beta}}{\omega - l\omega_H - k_z v_z}. \tag{27}$$

We can now make use of the asymptotic expression ($n \gg 1$) [17]

$$I_{mn}\left(\frac{a^2 k_\perp^2}{2}\right) \cong J_l(ak_\perp \sqrt{2n}) = J_l\left(\frac{k_\perp v_\perp}{\omega_H}\right)$$

to cast Equations (21) in the following form

$$\frac{\hbar}{M} t_{\alpha\beta} = \begin{pmatrix} v_\perp^2 J_l'^2 & -iv_\perp^2 \left(\frac{lJ_l}{\varrho}\right) J_l' & -iv_\perp v_z J_l J_l' \\ iv_\perp^2 \left(\frac{lJ_l}{\varrho}\right) J_l' & v_\perp^2 \left(\frac{lJ_l}{\varrho}\right)^2 & v_\perp v_z \left(\frac{lJ_l}{\varrho}\right) J_l \\ iv_\perp v_z J_l J_l' & v_\perp v_z \left(\frac{lJ_l}{\varrho}\right) J_l & v_z^2 J_l^2 \end{pmatrix}, \tag{27a}$$

where $\varrho \equiv k_\perp v_\perp/\omega_H$, $J_l \equiv J_l(\varrho)$, $J_l' \equiv \mathrm{d}J_l/\mathrm{d}\varrho$. Equations (27) and (27a) are identical with the well-known expressions obtained in the classical kinetic theory [18]. Our results differ from Sitenko's in so far as the x and y components of the tensor are interchanged. This is consistent with or choice of coordinates so that the propagation vector $\mathbf{k}$ is in the yz-plane.

In deriving Equation (27a) we have also made use of the identities $J_{l-1} - J_{l+1} = 2J_l'$ and $J_{l-1} + J_{l+1} = 2(lJ_l/\varrho)$ to write ($n \gg 1$)

$$\begin{aligned} &\left(\frac{\hbar\omega_H}{2M}\right)^{1/2} I_{mn}^{(-)} \cong -v_\perp J_l', \\ &\left(\frac{\hbar\omega_H}{2M}\right)^{1/2} \left(I_{mn}^{(+)} + \frac{ak_\perp}{\sqrt{2}} I_{mn}\right) \cong v_\perp \left(\frac{lJ_l}{\varrho}\right)\left[1 + \frac{a^2 k_\perp^2}{2l}\right]. \end{aligned} \tag{28}$$

The second term in the bracket of Equation (28) has been omitted. For strong magnetic fields, this is an excellent approximation except for very short wavelengths. Numerically $a^2/2 = \hbar/2M\omega_H \sim 3.3 \times 10^{-8} H^{-1}$ cm^2, where H is the magnetic field in gauss.

For long wavelengths when $\varrho \equiv k_\perp v_\perp/\omega_H \ll 1$ (numerically, $v_\perp/\omega_H \cong 1.7 \times 10^3 (v_\perp/c) \times \times H^{-1}$ cm), one has $J_l(\varrho) \cong J_l(0) = \delta_{l,0}$. In this case, the tensor $t_{\alpha\beta}$ becomes independent of the direction of propagation.

For the absorptive part our equations give in this limit

$$\tau^A_{\alpha\beta} = - i\pi \frac{2M^3}{N(2\pi\hbar)^3} \sum_{l=-\infty}^{\infty} \int \mathrm{d}^3 v \left(\frac{l\omega_H}{v_\perp} \frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z} \right) \frac{\hbar}{M} t_{\alpha\beta} \times \\ \times \delta(\omega - l\omega_H - k_z v_z),$$

which coincides with the classical prediction for the Landau damping.

B. DEGENERATE LIMIT

This limit is of particular theoretical interest since the description of a many-body system of interacting electrons in terms of a single-particle Fermi spectrum (quasi-particles) is known to be very good [19] (Throughout the derivation of Equation (13) we have implicitly assumed that only single-particle intermediate states are important).

In this limit, the distribution function is $f(E) = \theta(E_F - E)$. A Fermi momentum is defined for each Landau level as

$$p_F^{(n)} = \sqrt{2ME_F} \left(1 - \frac{n\hbar\omega_H}{E_F} \right)^{1/2}, \tag{29}$$

and Equation (18) becomes

$$N = \frac{2M\omega_H}{(2\pi\hbar)^2} \sum_n a_n p_F^{(n)} \equiv \frac{M\omega_H}{(2\pi\hbar)^2} P_0 . \tag{30}$$

The integrals in Equation (24) can be evaluated exactly, and one finds

$$\begin{aligned} \tau_\pm &= \sum_n a_n \tau_\pm^{(n)}, \\ \tau_\pm^{(n)} &\equiv \mp \frac{M\omega_H}{kP_0} \left[\log \left| \frac{M\omega_\pm - k(p_F^{(n)} \mp \frac{1}{2}\hbar k)}{M\omega_\pm + k(p_F^{(n)} \pm \frac{1}{2}\hbar k)} \right| + \right. \\ &\qquad \left. + n \log \left| \frac{M^2\omega_\pm^2 - k^2 (p_F^{(n)} \mp \frac{1}{2}\hbar k)^2}{M^2\omega_\pm^2 - k^2 (p_F^{(n)} \pm \frac{1}{2}\hbar k)^2} \right| \right], \\ \tau_{zz} &= 1 + \frac{M^2\omega^2}{\hbar k^3 P_0} \sum_n a_n \log \left| \frac{\omega^2 M^2 - k^2 (p_F^{(n)} + \frac{1}{2}\hbar k)^2}{\omega^2 M^2 - k^2 (p_F^{(n)} - \frac{1}{2}\hbar k)} \right|, \end{aligned} \tag{31}$$

where P_0 is defined by Equation (30).

In the limit $\hbar k / 2p_F^{(n)} \ll 1$, it is straight forward to see that [20]

$$\tau_\pm^{(n)} \cong \frac{M}{k} \log \left| \frac{M\omega_\pm - kp_F^{(n)}}{M\omega_\pm + kp_F^{(n)}} \right| \pm (2n+1) \frac{\hbar M k^2 p_F^{(n)}}{M^2\omega_\pm^2 - k^2 p_F^{(n)2}} . \tag{31a}$$

A general and more systematic study of the long wavelength limit is reserved for the next paragraph.

For the absorptive part we find from (25a)

$$\tau_{\pm}^{A} = i\pi \frac{M\omega_H}{P_0 k} \sum_{n=n_F-n_\omega}^{n_F} a_n \begin{Bmatrix} n \\ n+1 \end{Bmatrix} \tag{32}$$

where $n_F \equiv E_F/\hbar\omega_H$, and $n_\omega \cong \omega/\omega_H$ gives the number of occupied levels which can be excited above the Fermi surface by the radiation $\hbar\omega$.

7. Long Wavelength Limit Magneto-Ionic Theory

We first remark that for $a^2k_\perp^2/2 \ll n^{-1}$, $I_{mn}(a^2k_\perp^2/2) \cong I_{mn}(0) = \delta_{mn}$ and therefore Equations (21) approach the values of Equations (23) in this limit. The tensor $\tau_{\alpha\beta}$ then approaches the values given by Equations (24) and (25a) if we substitute k_z for k.

We now look for the limit of these equations when $k_z \to 0$. The bracket in Equation (24a) may be written as

$$\frac{\pm(2n+1)\dfrac{\hbar k^2}{2M} - \left(\omega_\pm - \dfrac{pk}{M}\right)}{\left(\omega_\pm - \dfrac{pk}{M}\right)^2 - \dfrac{\hbar^2 k^4}{4M^2}}.$$

Dropping the k^4 term in the denominator we obtain

$$\tau_\pm = \mp \frac{\omega_H}{P_0} \sum_n a_n \int_{-\infty}^{\infty} \mathrm{d}p\, f(E_{np}) \left[\frac{-1}{\omega_\pm - pk/M} \pm \frac{(2n+1)(\hbar k^2/2M)}{(\omega_\pm - pk/M)^2}\right] \tag{33}$$

and, similarly,

$$\tau_{zz} = -\frac{2k^2}{\omega^2 M^2 P_0} \sum_n a_n \int_{-\infty}^{\infty} \mathrm{d}p\, f(E_{np}) \left(p + \tfrac{1}{2}\hbar k\right)^3 . \tag{33a}$$

These expressions approach their $k=0$ values provided that

$$\begin{aligned} &\text{(a)} \quad \langle |p| \rangle \frac{k}{M} \ll \omega, \omega_\pm , \\ &\text{(b)} \quad \frac{\hbar k^2}{2M} \langle 2n+1 \rangle = \frac{\langle \pi_\perp^2 \rangle k^2}{2M^2 \omega_H} \ll \omega_\pm , \\ &\text{(c)} \quad k \ll p/\hbar , \end{aligned} \tag{34}$$

Correspondingly, one finds in this limit

$$\tau_\pm \cong \pm \frac{\omega_H}{\omega_\pm}; \qquad \tau_{zz} \cong -\frac{3k^2}{\omega^2} \frac{\langle p^2 \rangle}{M^2} \ll 1 \tag{35}$$

or, by Equation (13),

$$\varepsilon_{\pm} \cong 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_H)}, \qquad \varepsilon_{zz} \cong 1 - \frac{\omega_p^2}{\omega^2}. \tag{36}$$

These results are identical with the predictions of the classical magnetoionic theory [8]. It is noteworthy that Equations (36) were derived independently of the form of the distribution function $f(E_{np})$, and therefore are expected to hold equally well for equilibrium or nonequilibrium situations.

Equations (34) may be used to check whether the classical results, Equations (35), are applicable in cases of physical interest. Should Equations (35) give indices of refraction kc/ω inconsistent with Equations (34), one would have to use a refined theory based on Equations (24) and (25).

8. The One-Dimensional Gas

A. GENERAL CONSIDERATIONS

At the superstrong magnetic fields which are probably associated with neutron stars, an interesting situation arises, when the characteristic energy of the electrons is lower than the excitation energy of the Landau levels ($p^2/2M \ll \hbar\omega_H$). Only the lowest $n=0$ level is then populated, and the mobility of the electrons is therefore entirely determined by the value of p_z, thus giving rise to a one-dimensional electron gas. Low density, as well as an intense magnetic field, is necessary for this situation to be realized, since at densities $N \gtrsim 10^{28} \times H_{12}{}^{3/2}\ \mathrm{cm}^{-3}$, where $H_{12} = H/10^{12}$ G, the Fermi energy of the electrons becomes as high as $\hbar\omega_H$. Electron densities and fields satisfying this condition may arise in the plasma, which forms the atmosphere of a neutron star.

Retaining only the $n=0$ term in Equations (24), we find after a partial integration

$$\begin{aligned} \tau_{\pm} &= \pm \frac{M\omega_H}{kP_0} \int_{-\infty}^{\infty} \mathrm{d}p \, \frac{\partial f}{\partial p} \log|p \mp \tfrac{1}{2}\hbar k - \omega_{\pm} M/k|\,, \\ \tau_{zz} &= 1 - \frac{\omega^2 M^2}{\hbar k^3 P_0} \int_{-\infty}^{\infty} \mathrm{d}p \, \frac{\partial f}{\partial p} \log|(p + \tfrac{1}{2}\hbar k)^2 - \omega^2 M^2/k^2|\,, \end{aligned} \tag{37}$$

for the absorptive part of $\tau_{\alpha\beta}$ Equation (25a) give

$$\begin{aligned} \tau_{-}^{A} &= i\pi \frac{M\omega_H}{kP_0} [f(E_{p_-}) - f(E_{p_-} + \hbar\omega)]\,, \\ \tau_{zz}^{A} &= i\pi \frac{M^2\omega^2}{\hbar k^3 P_0} [f(E_{p_o}) - f(E_{p_o} + \hbar\omega)]\,; \end{aligned}$$

while the component $\tau_{\perp}^{A}$ vanishes in this case. In the case of a degenerate electron plasma these expressions reduce to the $n=0$ term of Equation (31). We shall next consider the degenerate case as an example, which can be treated analytically. We will establish the domain of validity of the magnetoionic theory for this case, and discuss the properties of $\varepsilon_{\alpha\beta}$ outside this domain.

B. DOMAINS OF APPROXIMATION

It will be convenient for our discussion to introduce the following dimentionless parameters $A = pk/M\omega$, $B = pk/M\omega_H$, $B_\pm = pk/M\omega_\pm$, $C = \hbar k/p$. p, here, represents a characteristic value of the electron's momentum in the z-direction. According to Equations (34) these parameters must be small compared to unity for the magnetoionic theory to be applicable.

Writing $\tau_{\alpha\beta}$ in terms of these parameters we obtain for the degenerate case

$$\begin{aligned} \tau_\pm &= \tfrac{1}{4}B^{-1}\log\left|\frac{\tfrac{1}{2}C \pm B_\pm^{-1} + 2}{\tfrac{1}{2}C \pm B_\pm^{-1} - 2}\right|, \\ \tau_{zz} &= 1 + \tfrac{1}{4}A^{-2}C^{-1}\log\left|\frac{A^{-2} - (\tfrac{1}{2}C + 2)^2}{A^{-2} - (\tfrac{1}{2}C - 2)^2}\right|. \end{aligned} \tag{38}$$

Numerically we find $\langle |p| \rangle = \tfrac{1}{2}p_F \cong 2.51 \times 10^{-2} \times N_{27} \times Mc$, $\omega_H = eH/Mc \cong$ $\cong 1.76 \times 10^{19}\ \mathrm{s}^{-1}$,

$$\begin{aligned} A &= \frac{1}{2}\frac{p_F}{Mc}\frac{kc}{\omega} \cong 2.51 \times 10^{-2} \times n \times N_{27} \times H_{12}^{-1}, \\ B &= A\frac{\omega}{\omega_H} \cong 1.43 \times 10^{-11} \times n \times \omega_{10} \times N_{27} \times H_{12}^{-2}, \\ C &= \frac{100}{2.51}\frac{\hbar}{Mc}k \cong 5.13 \times 10^{-10} \times n \times \omega_{10} \times N_{27}^{-1} \times H_{12}, \end{aligned} \tag{39}$$

Where $n = kc/\omega$ stands for the index of refraction, and $\omega_{10} = \omega/10^{10}\ \mathrm{s}^{-1}$.

The domains for the various approximations are summarized in Figure 1. Two such domains are defined for the transverse components $\tau_\pm$. In the long wavelength regime B, $C \ll 1$ the magnetoionic expressions are valid, and the corresponding transverse plasmons are given by the classical dispersion relation $n_\pm^2 = 1 - \omega_p^2/\omega\ (\omega \pm \omega_H)$. In the short wavelength region B, $C \gg 1$, however, one finds $n_\pm^2 \cong 1 - \omega_p^2/\omega^2$, a result, which is independent of the magnetic field. Correspondingly four separate domains of approximation are defined for τ_{zz}. Notably in Region II (Figure 1) while $\tau_\pm$ are still given by the M. I. theory the longitudinal component τ_{zz} is not. In fact τ_{zz} becomes quite singular near the separation line of Regions I and II.

The above considerations as well as the general expressions Equation (37) are valid for propagation along the magnetic lines of force ($\theta = 0$). In any other direction of propagation $\tau_{\alpha\beta}$ depends on the angle θ through the tensor $t_{\alpha\beta}$ cf. Equation (21). This tensor, however, being a manifestation of the details of the Landau orbits of the electron, becomes essentially independent of the angle, when the wavelenght is long compared to the Larmor radius i.e. when $ak \ll 1$. It is thus concluded, that the region $C < 1$ (which includes both Regions I and II in Figure 1) is free of this dependence.

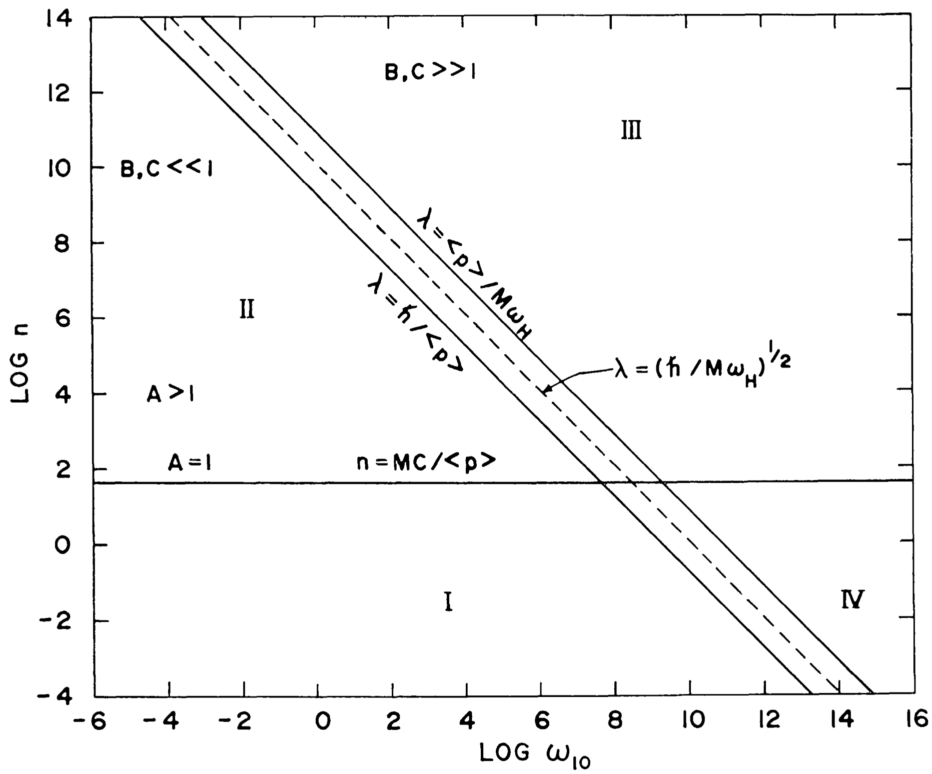

Fig. 1. The (n, ω) diagram showing the regions of the various approximations. The magnetoionic theory is valid in Region I (small n and long wavelengths). A simply modified version of the M. I. theory was shown to apply throughout the region of long wavelengths embracing both Regions I and II. When the wavelengths become shorter than the Larmor radius $a = (\hbar/M\omega_H)^{1/2}$, details of the Landau orbits may become important. Here $\omega_H \simeq 1.76 \times 10^{19}\ \mathrm{s}^{-1}$ and $\langle p \rangle \simeq 2.5 \times 10^{-2}\ Mc$.

The only remaining dependence of the angle is found by replacing $k_z = k \cdot \cos\theta$ instead of k in Equations (39). The dependence on the structure of the Landau orbits is expected to become important in the short wavelength Regions III and IV.

Before the foregoing analysis can be applied to the atmosphere of neutron stars, the effect of the ions on $\tau_{\alpha\beta}$, and the damping due to collisions would have to be considered. The ions contribute an additive term $\tau^i_{\alpha\beta}$, which is given by the classical kinetic theory, so that $\tau_{\alpha\beta} = \tau^i_{\alpha\beta} + \tau^e_{\alpha\beta}$ with $\tau^e_{\alpha\beta}$ as given by Equation (37). From the classical theory one knows, that the effect of the ions is an important one at low frequencies $\omega < (Z_i/A_i)\omega_H/2000$ [1, 7], where Z_i, A_i stand for the atomic number and atomic weight of the ion. The application to the atmosphere of a neutron star will be reserved for a future communication.

9. Collisions

The effect of collisions can be formally included in our discussion by introducing an effective collision frequency ν through the substitution $\omega \rightarrow \omega + i\nu$ in the denominators of Equations (20). An estimate for ν can be obtained in the relaxation time approximation as $v_j = \sigma v N_j$, where N_j is the density of heavy scatterers of type j, σ the collision cross section, and v the electron velocity.

For collisions with atoms and molecules σ is usually taken to be independent of the velocity v and roughly equal to the geometric cross section πr_0^2 where r_0 is the atomic (or molecular) radius. For collisions with ions, neglecting the effect of the magnetic field one obtains [7, 21]

$$\sigma_i = 2\pi \left(\frac{Z_i e^2}{mv^2}\right)^2 \log\left(1 + \cot^2 \tfrac{1}{2}\theta_0\right).$$

The minimum angle of scattering θ_0 is introduced because of the screening, and is related to the ratio between the electron's closest approach, and the screening radius. This expression is accurate for $\theta_0 \ll 1$, otherwise a more detailed computation of the screening effect is necessary, and multiple scattering also becomes important.

The presence of the intense magnetic field, however, may considerably modify these results. One knows, for instance [22], that in field of the order of 10^{12} G, the atom takes a strongly deformed shape, due to the magnetic constriction of the electronic orbits, at the same time becoming much more compact than the Bohr atom. One may therefore expect its effective cross section to decrease. Similarly the cross section for the Coulomb scattering may be affected by the magnetic field as well as the screening effect [23]. The quantum mechanical treatment of the Coulomb scattering in the magnetic field will be taken up in a future communication.

10. Conclusions

The dielectric tensor for a nonrelativistic electron gas imbedded in a constant and homogeneous magnetic field was computed within the framework of the quantum mechanical perturbation theory. Many-body effects and the Fermi degeneracy were taken into account within the accuracy of the random-phase approximation. We have used a one-photon linearized approach, and therefore our results do not apply to the propagation of intense electromagnetic radiation, where nonlinear effects become important.

Our computation is otherwise completely general. Going to the various limits we have recovered a number of previously known expressions. We finally gave a detailed and general discussion of the important case of the one-dimensional gas, which presumably finds application in the atmosphere of magnetic neutron stars.

As shown in detail in the main text, the quantum treatment of $\varepsilon_{\alpha\beta}$ does not preclude the use of the magnetoionic formulae, but merely defines the boundary of their validity. The extent of this boundary was discussed numerically in the case of the one-dimensional gas.

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